The Calabi invariant for some groups of homeomorphisms

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Abstract

We show that the Calabi homomorphism extends to some groups of homeomorphisms on exact symplectic manifolds.

The proof is based on the uniqueness of the generating Hamiltonian (proved by Viterbo) of continuous Hamiltonian isotopies (introduced by Müller and Oh).

1 Introduction

1.1 The Calabi homomorphism

Let (M, ω) be a symplectic manifold, supposed to be *exact*, that is $\omega = d\lambda$ for some 1-form λ called *Liouville form*. Equivalently, this also means that there exists a vector field X such that the Lie derivative satisfies: $\mathcal{L}_X\omega = \omega$. The vector field X is called the *Liouville vector field* and is related to the 1-form λ by the relation $\iota_X\omega = \lambda$. For instance, cotangent bundles are exact symplectic manifolds.

Thanks to the work of Banyaga [1, 2], the algebraic structure of the group $\mathrm{Ham}_c(M, \omega)$ of smooth compactly supported Hamiltonian diffeomorphisms of (M, ω) is quite well understood: there exists a group homomorphism, defined by Calabi [3]

$$\mathrm{Cal} : \mathrm{Ham}_c(M, \omega) \to \mathbb{R},$$

whose kernel ker(Cal) is a simple group.

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The Calabi homomorphism is defined as follows. Let $\phi \in \mathrm{Ham}_c(M, \omega)$ and let H be a compactly supported Hamiltonian function generating ϕ, i.e., a smooth function $[0,1] \times M \to \mathbb{R}$ such that:

- ϕ is the time one map of the flow $(\phi_H^t)_{t\in[0,1]}$ of the only time dependent vector field X_H satisfying at any time $t \in [0,1]$,

$$\iota_{X_H(t,\cdot)}\omega = dH(t,\cdot),$$

- there exists a compact set in M that contains all the supports of the functions $H_t = H(t,\cdot)$, for $t \in [0,1]$.

Then, by definition,

$$\mathrm{Cal}(\phi) = \int_0^1 \int_M H(t,x)\omega^d dt, \tag{1}$$

where d is half the dimension of M. This expression does not depend on the choice of the generating function H, and gives a group homomorphism.

1.2 Question and result

We consider the following question.

Question 1.2.1. *To which groups of homeomorphisms does the Calabi homomorphism extend?*

Note that the Calabi homomorphism does not behave continuously with respect to the C^0-topology, as shows the following example.

Example 1.2.2. Let $\phi \in \mathrm{Ham}_c(\mathbb{R}^2, rdr \wedge d\theta)$, and consider the sequence (ϕ_n) in $\mathrm{Ham}_c(\mathbb{R}^2, rdr \wedge d\theta)$ given by

$$\phi_n(r,\theta) = \frac{1}{n}\phi^{4n}(nr,\theta).$$

This sequence converges in the C^0-sense to Id, but one can easily check that its Calabi invariant remains constant.

We will define three interesting groups of homeomorphisms, denoted G_1, G_2 and G_3, and prove the following result.

Theorem 1.2.3. *The Calabi homomorphism extends to a group homomorphism $G_3 \to \mathbb{R}$. Moreover, we have the following inclusions $\mathrm{Ham}_c(M,\omega) \subset G_1 \subset G_2 \subset G_3$.*

We will give the full definitions of G_1, G_2 and G_3 in Section 2. Let us still give here an idea of what they are:

G_1 is the identity component of the group of compactly supported symplectic bilipschitz homeomorphisms whose flux is zero (see Section 2.1).

G_2 is the group generated by the homeomorphisms that admit particular generating functions (see Section 2.2).

G_3 is the group of homeomorphisms ϕ such that (on some interval where it is well defined) the isotopy $t \mapsto [\mu_t, \phi]$ is a C^0-Hamiltonian isotopy (in the sense of [9]). Here, μ_t denotes the flow generated by the Liouville vector field X, and $[\mu_t, \phi] = \mu_t \circ \phi \circ \mu_t^{-1} \circ \phi^{-1}$ (see Section 2.3).

REMARK 1.2.4. In the special case of the (2-dimensional) open disk, the fact that the Calabi homomorphism extends to G_1 was already proved by Haissinsky [6][1]. His methods are completely different.

Let us also mention that Gambaudo and Ghys have proved that two diffeomorphisms of the disk that are conjugated by an area preserving homeomorphism have same Calabi invariant [5].

1.3 Motivation

Our motivation for this work comes from two distinct problems. The first one comes from the following question which remains open.

Question 1.3.1 (Fathi [4]). *Is the group* $\mathrm{Homeo}_c(\mathbb{D}_2, area)$ *of compactly supported area preserving homeomorphisms of the disk a simple group ?*

Several non-trivial normal subgroups of $\mathrm{Homeo}_c(\mathbb{D}_2, area)$ have been defined by Ghys [2], Müller-Oh [10] and recently by Le Roux [8]. But so far, no one has been able to prove that any of them is a proper subgroup.

Our study is inspired by the work of Müller and Oh. They introduced on any symplectic manifold (M, ω) a group denoted $\mathrm{Hameo}(M, \omega)$, whose elements are homeomorphisms called *hameomorphisms* (as the contraction of "Hamiltonian homeomorphisms"). This group contains all compactly supported Hamiltonian diffeomorphisms and, in the case of the disk, forms a normal subgroup of $\mathrm{Homeo}_c(\mathbb{D}_2, area)$. A. Fathi noticed that if one could extend the Calabi homomorphism to the group of hameomorphisms, then it would be necessarily a proper subgroup, and $\mathrm{Homeo}_c(\mathbb{D}_2, area)$ would not be simple.

[1] Area preserving quasiconformal maps of the plane are bilipschitz. Therefore, Haissinsky's result is precisely the fact that the Calabi homomorphism extends to G_1.

In the present paper, we propose a different approach: instead of constructing a group which is known to be normal but on which it is unknown whether the Calabi homomorphism extends, we construct a group to which the Calabi invariant extends but for which it is unknown whether it is normal.

Another motivation is a very natural general problem: how can one generalize Hamiltonian dynamics in a non-smooth context? or (less optimistic) which properties of Hamiltonian maps can be extended? The present paper concentrates on a particular aspect: the Calabi homomorphism.

Our interest in the groups G_1 and G_2 comes from the fact that they give large families of examples of elements of G_3, but also from the fact they are quite natural generalizations of the Hamiltonian group, which could be considered to study the extension of other aspects of Hamiltonian dynamics. As an example, all the rigidity results obtained on Hamiltonian diffeomorphisms using generating functions technics may also hold for the elements of G_2 (and thus of G_1).

Several other possible groups generalizing the Hamiltonian group have already been considered in literature. The group $\mathrm{Hameo}(M, \omega)$ mentioned above is one of them, another has been studied by the author in [7]. But this direction of research is still to be developed.

2 The three groups

2.1 The group G_1

The group $\mathrm{Ham}_c(M, \omega)$ can be characterised as the set of all symplectic diffeomorphism which are compactly supported, isotopic to the identity and with zero flux. It is thus natural to introduce the following definition.

Definition 2.1.1. *We denote by G_1 the identity component of the group of compactly supported bilipschitz (for some Riemannian metric) symplectic homeomorphisms whose flux is zero.*

Remark 2.1.2. Since Lipschitz maps are almost everywhere differentiable, the pull-back of a differential form by a bilipschitz map is well-defined as a differentiable form with L^∞ coefficients. Therefore, as in the smooth case, a bilipschitz homeomorphism ϕ of M is symplectic if $\phi^*\omega = \omega$, and has zero flux if the one form $\lambda - \phi^*\lambda$ is exact (recall that (M, ω) is supposed to be exact, with Liouville form λ).

Note that a bilipschitz homeomorphism which is the C^0-limit of smooth symplectomorphisms is symplectic in this sense.

2.2 The group G_2

The group G_2 is another natural generalisation of the Hamiltonian group, based on the notion of generating function, that we describe now.

First recall that, according to Weinstein's Neighbourhood Theorem [12], there exists a neighbourhood U of the diagonal Δ in the symplectic manifold $(M \times M, \omega \oplus (-\omega))$, symplectomorphic to a neighbourhood of the zero section in the cotangent $T^*\Delta$ and hence to a neighbourhood V of the zero section in T^*M. We will denote $j : U \to V$ such a symplectomorphism, and call it a *Weinstein chart*.

Now, for any symplectic diffeomorphism ϕ of M, the image of its graph

$$L_\phi = j(\mathrm{graph}(\phi)) = j(\{(x, \phi(x)) \in M \times M \mid x \in M\})$$

is a lagrangian submanifold in T^*M. Moreover, L_ϕ is exact if and only if ϕ is Hamiltonian. If in addition ϕ is sufficiently close to the identity in the C^1-sense, L_ϕ is the graph of the differential of a smooth compactly supported function $S : M \to \mathbb{R}$:

$$L_\phi = \mathrm{graph}(dS) = \{(x, dS(x)) \in T^*M \mid x \in M\}.$$

We then say that ϕ admits S as a generating function.

Since any Hamiltonian diffeomorphism can be written as a product of Hamiltonian diffeomorphisms C^1-close to the identity, the group $Ham_c(M, \omega)$ can be characterized as the subgroup of the diffeomorphisms of M generated by the elements that admit smooth compactly supported generating functions. When one tries to extend a property of Hamiltonian diffeomorphisms to homeomorphisms (the Calabi invariant in our case), it is thus natural to consider homeomorphisms that admits generating functions. This idea leads to the following definition.

Definition 2.2.1. *A C^1 compactly supported function $S : M \to \mathbb{R}$ is called an* admissible generating function *if there exist a homeomorphism ϕ of M, and a symplectic diffeomorphism j between a neighbourhood U of the diagonal in $M \times M$ and a neighbourhood V of the zero section in T^*M such that*

- *$graph(\phi) \subset U$,*
- *$graph(dS) \subset V$,*
- *$j(graph(\phi)) = graph(dS)$.*

The homeomorphism ϕ associated to S is called an admissible homeomorphism.

An admissible generating function is called super-admissible *if it is the limit in the C^1-sense of a sequence of C^∞ admissible generating functions. A* super-admissible homeomorphism *is an admissible homeomorphism associated to a super-admissible generating function.*

We denote by G_2 the group generated by homeomorphisms ϕ for which there exists some real number $\delta > 0$ such that for any t in $[0, \delta]$, the conjugation by the Liouville flow $\mu_t \circ \phi \circ \mu_t^{-1}$ is also super-admissible.

Remark 2.2.2. As in the introduction, $\mu_t(x)$ denotes the flow (when it is defined) of the Liouville vector field X, at time t and point $x \in M$. Note that it satisfies $\mu_t^* \omega = e^t \omega$.

Let ϕ be a compactly supported homeomorphism of M. Then there exists a real number $\delta > 0$, such that for any $t \in [0, \delta]$, μ^t and $(\mu^t)^{-1}$ are well defined on the support of ϕ. Thus, the conjugation $\mu_t \circ \phi \circ \mu_t^{-1}$ is well defined on $\mu_t(\mathrm{Supp}(\phi))$. In the complement of this set, it is the identity where it is defined. Therefore, we can extend it to a well defined homeomorphism still denoted $\mu_t \circ \phi \circ \mu_t^{-1}$ just by setting it to equal the identity where it is not defined.

Even though the definition of G_2 looks quite strange, it is quite a large group. Indeed, it contains the group G_1, as stated in Theorem 1.2.3, and it also contains a large family of examples, that we shall construct in section 4.

2.3 The group G_3

To define the group G_3 we first need the following notion.

Definition 2.3.1 (Müller-Oh [10]). *A* C^0-Hamiltonian isotopy *is a path $(\phi^t)_{t \in [0,\delta]}$ of homeomorphisms of M for which there exist a compact set K and a sequence of smooth Hamiltonian functions H_n on M with support in K, such that*

- *(H_n) converges to some continuous function $H : [0, \delta] \times M \to \mathbb{R}$ in the C^0-sense,*
- *$(\phi^t_{H_n})$ converges to ϕ^t in the C^0-sense, uniformly in $t \in [0, \delta]$.*

The function H is called a C^0-Hamiltonian function *generating (ϕ^t).*

Remark 2.3.2. The elements of C^0-Hamiltonian isotopies are *symplectic homeomorphisms*, i.e., homeomorphisms which are the C^0 limit of a sequence of symplectic diffeomorphisms supported in a common compact set.

It is not difficult to check that if (ϕ^t) and (ψ^t) are two C^0-Hamiltonian isotopies generated by F and G, then $((\phi^t)^{-1})$ and $(\phi^t \circ \psi^t)$ are C^0-Hamiltonian isotopies generated by $-F(t,(\phi^t)^{-1}(x))$ and $F(t,x) + G(t,\phi^t(x))$, and that if f is any symplectic homeomorphism, $(f^{-1} \circ \phi^t \circ f)$ is a C^0-Hamiltonian isotopy generated by $F(t, f(x))$. This means that the computations are the same as in the smooth case.

The main result concerning C^0-Hamiltonian isotopies is:

Theorem 2.3.3 (Viterbo [11]). *A given C^0-Hamiltonian isotopy is generated by a unique C^0-Hamiltonian function.*

This theorem is the only non-trivial result needed in this paper. Its proof needs at some point a (hard!) rigidity result in symplectic topology due to Gromov.

By Remark 2.2.2, for any compactly supported homeomorphism ϕ the commutator

$$[\mu_t, \phi] = \mu_t \circ \phi \circ \mu_t^{-1} \circ \phi^{-1},$$

is well defined, for t small enough.

Definition 2.3.4. *We denote by G_3 the set of all compactly supported symplectic homeomorphisms ϕ for which there exists some $\delta > 0$ small enough, such that the isotopy $([\mu_t, \phi])_{t\in[0,\delta]}$ is a C^0-Hamiltonian isotopy.*

Clearly, G_3 contains $Ham_c(M,\omega)$.

Proposition 2.3.5. *The set G_3 is a group. Moreover, if the first compactly supported cohomology group $H^1_c(M,\mathbb{R})$ vanishes, G_3 does not depend on the choice of the Liouville vector field.*

Proof . — Let $\phi, \psi \in G_3$. For δ small enough $([\mu_t, \phi])_{t\in[0,\delta]}$ and $([\mu_t, \psi])_{t\in[0,\delta]}$ are C^0-Hamiltonian isotopies. Then, note that

$$[\mu_t, \phi \circ \psi] = [\mu_t, \phi] \circ (\phi \circ [\mu_t, \psi] \circ \phi^{-1}),$$

and

$$[\mu_t, \phi^{-1}] = \phi^{-1} \circ [\mu_t, \phi]^{-1} \circ \phi.$$

We conclude with Remark 2.3.2 that G_3 is a group.

Suppose now that $H^1_c(M,\mathbb{R}) = 0$, and that μ'_t is the flow of another Liouville vector field. Then, $\eta_t = \mu'_t \circ \mu_t^{-1}$ is a smooth symplectic isotopy which is Hamiltonian since $H^1_c(M,\mathbb{R}) = 0$. Using once again Remark 2.3.2 and the identity

$$[\mu'_t, \phi] = \eta_t \circ [\mu_t, \phi] \circ (\phi \circ \eta_t^{-1} \circ \phi^{-1}), \tag{2}$$

we conclude that G_3 would be the same if it was defined with another Liouville vector field. □

3 Proof of the main theorem

3.1 Extension of the Calabi homomorphism

In this section, we prove that the Calabi homomorphism extends to G_3. Let us first give a new formula for the Calabi, for which we need to choose a Liouville form instead of choosing an isotopy.

Lemma 3.1.1. *Let $\phi \in \mathrm{Ham}_c(M, \omega)$ and let $H_{\lambda,\phi}$ be the generating Hamiltonian function of the smooth Hamiltonian isotopy $([\mu_t, \phi])$. Then,*

$$\mathrm{Cal}(\phi) = \frac{1}{d+1} \int_M H_{\lambda,\phi}(0,x)\omega^n.$$

Proof. First note that if ϕ is the time one map of some Hamiltonian function H, and if we suppose $\mu_\delta^{-1} \circ \phi \circ \mu_\delta$ to be well defined, then it can be generated by the Hamiltonian function $e^\delta H \circ \mu_\delta^{-1}$. After an easy change of variables in Equation (1), one gets

$$\mathrm{Cal}(\mu_\delta^{-1} \circ \phi \circ \mu_\delta) = e^{(d+1)\delta}\mathrm{Cal}(\phi),$$

where d is half the dimension of M. Thus,

$$\mathrm{Cal}([\mu_\delta, \phi]) = (e^{(d+1)\delta} - 1)\mathrm{Cal}(\phi).$$

Hence, applying formula (1) to $H_{\lambda,\phi}$,

$$\mathrm{Cal}(\phi) = \frac{1}{e^{(d+1)\delta} - 1} \int_0^\delta \int_M H_{\lambda,\phi}(t,x)\omega^n dt.$$

Now, letting δ converge to 0, we get the desired formula. □

Once this formula obtained, extending the Calabi homomorphism to G_3 is very easy, even though it relies on the "hard symplectic topology" uniqueness Theorem 2.3.3.

Proof. let $\phi \in G_3$ and let H be the **unique** C^0-Hamiltonian function generating $([\mu_t, \phi])_{t\in[0,\delta]}$ for some small δ. We set:

$$\widetilde{\mathrm{Cal}}(\phi) = \frac{1}{d+1} \int_M H(0,x)\omega^n.$$

By Lemma 3.1.1, $\widetilde{\mathrm{Cal}}$ coincide with Cal on $Ham_c(M, \omega)$. Moreover using Remark 2.3.2 and the formulas in the proof of Proposition 2.3.5, one checks easily that $\widetilde{\mathrm{Cal}} : G_3 \to \mathbb{R}$ is a group homomorphism. □

REMARK 3.1.2. If $H_c^1(M, \mathbb{R}) = 0$, then $\widetilde{\mathrm{Cal}}$ does not depend on the choice of the Liouville vector field. This is an immediate consequence of Equation 2.

3.2 Proof of the inclusion $G_1 \subset G_2$

We are going to prove that an element of G_1 which is sufficiently close to the identity in the bilipschitz sense is a super-admissible homeomorphism (Definition 2.2.1). Since any element g of G_1 can be written as a product of elements of G_1 close to the identity (simply cut any path joining g to the identity in small pieces), this will imply that G_1 is included in G_2. This fact is standard for diffeomorphisms, and is not more difficult in the bilipschitz case.

Let $g \in G_1$, close enough to the identity in the bilipschitz sense. Then, in particular g is C^0-close to the identity and its graph lies in the domain of a Weinstein chart $j : U \to V$. Now, the map $\mathrm{Id} \times g : M \to M \times M$, $x \mapsto (x, g(x))$ is Lipschitz close to the diagonal inclusion $x \mapsto (x, x)$. As a consequence, the conjugated map $a = j \circ (\mathrm{Id} \times g) \circ j^{-1}$ is Lipschitz-close to the zero section of the cotangent bundle T^*M. Standard arguments (the same as in the C^1 case) then show that the image of a is the graph of the section s of T^*M given by

$$s = a \circ (\pi \circ a)^{-1},$$

where $\pi : T^*M \to M$ is the canonical projection. Moreover, this section s is Lipschitz-close to the zero section.

It remains to prove that the Lipschitz 1-form s is exact. This follows from the fact that the flux of the homeomorphism g vanishes. Indeed, since g has zero flux, for any Liouville form λ, $(\mathrm{Id} \times g)^*(\lambda \oplus (-\lambda))$ is an exact one form on M. Since the map $\mathrm{Id} \times g$ is homotopic to the map $\mathrm{Id} \times \mathrm{Id}$, this implies that the pull-back of any primitive of $\omega \oplus (-\omega)$ is exact. Let λ_0 denotes the standard Liouville form on T^*M, one has $dj^*\lambda_0 = (\omega \oplus (-\omega))$ hence $(\mathrm{Id} \times g)^* j^* \lambda_0$ is exact. It follows that $s = s^*\lambda_0 = (q^{-1})^* a^* \lambda_0$ is exact.

Now, if we denote by S the compactly supported primitive of s, it is a $C^{1,1}$-function which is admissible by construction. Moreover, it is small in the $C^{1,1}$ sense and thus can be approximated in the C^1 sense by C^2-small smooth functions. But it is well known that C^2-small smooth functions are admissible. Therefore, S is super-admissible.

Finally, for t small enough, $\mu_t \circ g \circ \mu_t^{-1}$ remains Lipschitz-close to the identity. Thus, g is one of the generators of G_2. $\square$

3.3 Proof of the inclusion $G_2 \subset G_3$

Theorem 1.2.3 clearly follows from the following proposition. We denote by $\Psi(S)$ the admissible homeomorphism associated to an admissible generating function S.

Proposition 3.3.1. *Let $t \mapsto S_t$, $t \in [0,\delta]$ be a C^1 path of super-admissible generating functions, associated to a fixed Weinstein chart, which is the C^1-limit of a smooth path of smooth admissible generating functions. Then, the path $t \mapsto \Psi(S_t)$ is a C^0-Hamiltonian isotopy.*

Remark 3.3.2. We can construct examples of such paths using Darboux coordinates (see Section 4 below). By the way, this proposition gives new examples of C^0-Hamiltonian isotopies. As an example, the argument shows that any Lipschitz continuous path in G_1 is a C^0-Hamiltonian isotopy.

To prove Proposition 3.3.1, we will need two (classical) lemmas.

Lemma 3.3.3. *Let $j : U \to V$ be a Weinstein chart. For any integer $k \geqslant 0$, the map Ψ is a homeomorphism between the set of C^{k+1} admissible generating functions associated to j (endowed with the C^{k+1}-topology) and the set of C^k admissible (diffeo)homeomorphisms associated to j (endowed with the C^k-topology).*

Proof . — The details of the proof of this lemma will be left to the reader. We just give here the idea: as in the previous section, we use the relation between S and $\Psi(S)$. Denote $a = j \circ (\mathrm{Id} \times \Psi(S)) \circ j^{-1}$. Then by construction, $\pi \circ a$ is invertible and one has

$$dS = a \circ (\pi \circ a)^{-1}.$$

This gives continuity properties of Ψ^{-1}.

Conversely, if we consider $p_1 : M \times M \to M$ the projection on the first factor, and denote $b = j^{-1} \circ dS \circ j$, then by construction, $p_1 \circ b$ is invertible and one has

$$\mathrm{Id} \times \Psi(S) = b \circ (p_1 \circ b)^{-1}.$$

This allows to prove continuity properties for Ψ. □

Lemma 3.3.4. *Let $t \mapsto S_t$ be a smooth path of smooth admissible generating functions associated to a fixed Weinstein chart and denote H the compactly supported Hamiltonian function that generates the Hamiltonian isotopy $t \mapsto \Psi(S_t)$. Then,*

$$H(t,x) = -\frac{\partial S_t}{\partial t}(\pi \circ j \circ (\Psi(S_t)^{-1} \times \mathrm{Id}) \circ j^{-1}(x)).$$

In $\mathbb{R}^{2n}$, this formula is just the classical Hamilton-Jacobi Equation.

Proof . — We set $f_t = \mathrm{Id} \times \Psi(S_t)$, $q_t = \pi \circ j \circ f_t \circ j^{-1}$, and denote by λ_0 the canonical Liouville form on T^*M. We have seen in the proof of Lemma 3.3.3 that $dS_t \circ q_t \circ j = j \circ f_t$.

We first pull back the Liouville form. Since $\sigma^*\lambda_0 = \sigma$ for any 1-form σ on M, $j^* q_t^* dS_t^* \lambda_0 = j^* q_t^* dS_t = d(S_t \circ q_t \circ j)$. We thus have:

$$d(S_t \circ q_t \circ j) = f_t^*(j^*\lambda_0).$$

We then take derivative with respect to t:

$$\begin{aligned} & d\left(\frac{\partial S_t}{\partial t}(q_t \circ j) + dS_t(q_t \circ j) \cdot \frac{dq_t}{dt} \circ j\right) \\ & \quad = \; f_t^*(\iota_{\frac{df_t}{dt}\circ f_t^{-1}} d(j^*\lambda_0)) + d(f_t^*(\iota_{\frac{df_t}{dt}\circ f_t^{-1}}(j^*\lambda_0))). \end{aligned}$$

But since j is symplectic, $d(j^*\lambda_0) = \omega \oplus (-\omega)$ hence

$$f_t^*(\iota_{\frac{df_t}{dt}\circ f_t^{-1}} d(j^*\lambda_0)) = 0 - \Psi(S_t)^*(\iota_{\frac{\Psi(S_t)}{dt}\circ \Psi(S_t)^{-1}}\omega) = -d(H_t \circ \Psi(S_t)).$$

Therefore, after taking (compactly supported) primitive, we get:

$$\frac{\partial S_t}{\partial t}(q_t \circ j) + dS_t(q_t \circ j) \cdot \frac{dq_t}{dt} \circ j = -H_t \circ \Psi(S_t) + f_t^*(\iota_{\frac{df_t}{dt}\circ f_t^{-1}}(j^*\lambda_0)).$$

It remains to show that $dS_t(q_t \circ j) \cdot \frac{dq_t}{dt} \circ j = f_t^*(\iota_{\frac{df_t}{dt}\circ f_t^{-1}}(j^*\lambda_0))$. To see this, recall that for any one form σ on M, the pullback $\pi^*\sigma$ by the canonical projection coincides with λ_0 on the image of σ (which is a smooth submanifold of T^*M). Then,

$$\begin{aligned} dS_t(q_t \circ j) \cdot \frac{dq_t}{dt} \circ j &= (\pi^* dS_t)(j \circ f_t) \cdot dj \frac{df_t}{dt} \\ &= (j^*\lambda_0)(f_t) \cdot \frac{df_t}{dt}. \end{aligned}$$

This concludes our proof. $\square$

Proof of Proposition 3.3.1. — Let (S_t) be our path of generating functions. By assumption, there is a sequence of smooth paths of smooth admissible generating functions (S_t^k) that converges in the C^1-sense to (S_t). Let H_k be the generating Hamiltonian function of the Hamiltonian isotopy $\Psi(S_t^k)$.

By Lemma 3.3.3, the isotopies $(\phi_{H_k}^t) = (\Psi(S_t^k))$ C^0-converge to $\Psi(S_t)$. Moreover, by Lemma 3.3.4, the Hamiltonian functions

$$H_k = \frac{\partial S_t^k}{\partial t}(\pi \circ j \circ (\Psi(S_t^k)^{-1} \times \mathrm{Id}) \circ j^{-1}(x))$$

also C^0-converge. This shows that $(\Psi(S_t))$ is a C^0-Hamiltonian isotopy. $\square$

4 Examples in $\mathbb{R}^{2n}$

In this section, we give some examples of elements in G_2 and G_3 in $\mathbb{R}^{2n}$. Using local Darboux coordinates, they can of course be implanted in other symplectic manifolds.

4.1 Examples of elements in G_2

In $\mathbb{R}^{2n} \times \mathbb{R}^{2n}$, there exists globally defined Weinstein charts sending the diagonal to the zero section in $T^*\mathbb{R}^{2n}$. We will use the following one:

$$j : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to T^*\mathbb{R}^{2n} = \mathbb{R}^{2n} \times \mathbb{R}^{2n}, (x, y; \xi, \eta) \mapsto (x, \eta; y - \eta, \xi - x).$$

In this Weinstein chart, admissible homeomorphisms and admissible generating functions are associated by the following relation:

$$f(x, y) = (\xi, \eta) \quad \Longleftrightarrow \quad \begin{cases} \xi = x + \frac{\partial S}{\partial \eta}(x, \eta) \\ y = \eta + \frac{\partial S}{\partial x}(x, \eta) \end{cases} .$$

Therefore, admissible generating function are the compactly supported C^1 functions $S : \mathbb{R}^{2n} \to \mathbb{R}$ such that

- for all $\eta \in \mathbb{R}^n$, the map $x \mapsto x + \frac{\partial S}{\partial \eta}(x, \eta)$ is a homeomorphism of $\mathbb{R}^n$,
- for all $x \in \mathbb{R}^n$, the map $\eta \mapsto \eta + \frac{\partial S}{\partial x}(x, \eta)$ is a homeomorphism of $\mathbb{R}^n$.

Proposition 4.1.1. *Any compactly supported C^1 function $S : \mathbb{R}^{2n} \to \mathbb{R}$ such that, in any point $(x, \eta) \in \mathbb{R}^{2n}$ the maps*

$$x_i \mapsto x_i + \frac{\partial S}{\partial \eta_i}(x, \eta) \text{ and } \eta_i \mapsto \eta_i + \frac{\partial S}{\partial x_i}(x, \eta), \text{ for } i \in \{1, \dots, n\},$$

are increasing homeomorphisms of $\mathbb{R}$, is a super-admissible generating function.

Proof . — First, such a function is admissible: for any $x, \eta \in \mathbb{R}^n$ the maps $\eta \mapsto \eta + \frac{\partial S}{\partial x}(x,\eta)$ and $x \mapsto x + \frac{\partial S}{\partial \eta}(x,\eta)$ are homeomorphisms of $\mathbb{R}^n$.

Indeed, one see easily that $\eta \mapsto \eta + \frac{\partial S}{\partial x}(x,\eta)$ is continuous and injective. Since it is compactly supported, it is also proper and hence is an embedding. Finally, this implies that it is onto, because otherwise its image would contain non-contractible spheres $\mathbb{S}_{n-1}$. The same argument holds for $x \mapsto x + \frac{\partial S}{\partial \eta}(x,\eta)$.

Let us now show that S can be approximated in the C^1-sense by smooth generating functions.

Let χ be a smooth non-negative function, defined on $\mathbb{R}^{2n}$, whose support is contained in a disk centered in 0 and with integral equal to 1. For any positive integer k, we set $\chi_k = k^{2n}\chi(\frac{\cdot}{k})$. Then, it is well known that the sequence of smooth functions (S_k) defined by

$$S_k(x,\eta) = \chi_k * S(x,\eta) = \int_{\mathbb{R}^{2n}} S(x-u,\eta-v)\chi_k(u,v)\,du\,dv,$$

C^1-converges to S as k goes to infinity. Moreover, there exists a compact set that contains the supports of every S_k.

let us now prove that the S_k are admissible generating functions. Set

$$\alpha(x,\eta) = x + \frac{\partial S}{\partial \eta}(x,\eta) \text{ and } \beta(x,\eta) = \eta + \frac{\partial S}{\partial x}(x,\eta).$$

According to the first part of the proof, it is enough to prove that for any indices i, the maps $x_i \mapsto q_i \circ (\chi_k * \alpha(x,\eta))$ and $\eta_i \mapsto p_i \circ (\chi_k * \beta(x,\eta))$ are increasing homeomorphisms of $\mathbb{R}$. They are clearly continuous. Since they are compactly supported, we only need to show that they are increasing. Let us prove it for $x_1 \mapsto q_1 \circ (\chi_k * \alpha(x,\eta))$. The proof is similar for the others.

Fix $\eta, x_2, \ldots, x_n$ and $x_1 < x_1'$ and denote $x = (x_1, x_2, \ldots, x_n)$ and $x' = (x_1', x_2, \ldots, x_n)$. We want to compare $q_1 \circ (\chi_k * \alpha(x,\eta))$ with $q_1 \circ (\chi_k * \alpha(x',\eta))$. By assumption, for all $(u,v) \in \mathbb{R}^{2n}$,

$$q_1 \circ \alpha(x-u,\eta-v) \; < \; q_1 \circ \alpha(x'-u,\eta-v),$$

thus the following integral is non-negative:

$$\int_{R^{2n}} \chi_k(u,v) \left[\, q_1 \circ \alpha(x'-u,\eta-v) - q_1 \circ \alpha(x-u,\eta-v) \,\right] \, du\,dv.$$

It is moreover positive because it is the integral of a non-negative continuous function which is non-identically zero. This integral is nothing but $q_1 \circ (\chi_k * \alpha(x,\eta)) - q_1 \circ (\chi_k * \alpha(x',\eta))$. Therefore the map $x_1 \mapsto q_1 \circ (\chi_k * \alpha(x,\eta))$ is an increasing homeomorphism of $\mathbb{R}$. □

REMARK 4.1.2. The conjugation $\mu_t \circ \phi \circ \mu_t^{-1}$ by the Liouville flow $\mu_t : x \mapsto e^{t/2}x$ of an homeomorphism ϕ associated to a generating function S like in Proposition 4.1.1, is also admissible and is associated to the generating function $e^t S(e^{-t/2}x, e^{-t/2}\eta)$. This function satisfies the hypothesis of Proposition 4.1.1 and hence is also a super-admissible generating function. It follows that such a ϕ is in G_2.

REMARK 4.1.3. Any generating function like in Proposition 4.1.1, which is **not** $C^{1,1}$, gives rise to an example of element which is in G_2 but not in G_1.

4.2 Fibered rotations in $\mathbb{R}^2$

By definition a *fibered rotation* is an homeomorphism ϕ of $\mathbb{R}^2$ described in polar coordinates (r, θ) by the formula

$$\phi(r, \theta) = (r, \theta + \rho(r)),$$

for some continuous *angular* function $\rho : (0, +\infty) \to \mathbb{R}$ with bounded support. It is easily checked that any fibered rotation lies in the identity component of the group of compactly supported area preserving homeomorphism of $\mathbb{R}^2$.

We consider μ_t the Liouville flow given by $\mu_t(r, \theta) = (e^{t/2}r, \theta)$. Its commutator with a fibered rotation is given by

$$[\mu_t, \phi](r, \theta) = (r, \theta - \rho(r) + \rho(e^{-t/2}r)).$$

If ϕ is moreover a diffeomorphism, the generating Hamiltonian of the isotopy $t \mapsto [\mu_t, \phi]$ is

$$H(t, r, \theta) = r\rho(e^{-t/2}r) - \frac{1}{2}\int_0^r \rho(e^{-t/2}s)\, ds.$$

Now suppose that ρ is a continuous and integrable angular function, such that $r\rho(r)$ converges to 0 when r tends to 0. Suppose also that ρ_k is a sequence of smooth compactly supported angular functions (in particular they vanish nearby 0) that converges uniformly to ρ on any compact subset of $(0, +\infty)$. Then, clearly, the associated sequence of fibered rotations (ϕ_k) converges in the C^0-sense to ϕ, and the sequence of Hamiltonians (H_k) generating the isotopies $t \mapsto [\mu_t, \phi_k]$ also C^0-converges.

As a consequence, *any fibered rotation associated to an integrable angular function ρ such that $r\rho(r) \xrightarrow{r \to 0} 0$, belongs to G_3.*

REMARK 4.2.1. This gives examples of elements that are in G_3 but not in G_2: if ρ is not finite (nearby 0), the fibered rotation ϕ cannot

be in G_2. Indeed, the angle between a vector and its image by an admissible homeomorphism is bounded by π. Therefore, this angle has to be finite for elements of G_2.

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